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Abstract. We prove that cyclic languages are the boolean closure of languages called strongly cyclic languages. The result is used to give another proof of the rationality of the zeta function of rational cyclic languages.

1 Introduction

Cyclic languages and strongly cyclic languages are two classes of languages of finite words over a finite alphabet. A cyclic language is conjugation-closed and for any two words having a power in common, if one of them is in the language, then so is the other. A strongly cyclic language is the set of words stabilizing a subset of the set of states of a finite deterministic automaton, the subset stabilized depending on the word stabilizing it. One says that the language stabilizes the automaton. A strongly cyclic language is rational.

We prove that a rational cyclic language is a boolean combination of strongly cyclic languages. More precisely, each rational cyclic language can be written as a chain of strongly cyclic languages. This result allows us to extend the computation of the zeta function (and generalized zeta function) of strongly cyclic languages done in [Béa95] to the class of rational cyclic languages. The zeta function of a formal language L is $\zeta(L) = \exp(\sum a_n \frac{z^n}{n})$, where a_n is the number of words of length n in L. The motivations of this definition and the connections with algebraic geometry and symbolic dynamics are discussed in [BR90]. The zeta function of a strongly cyclic language L is equal to the zeta function of the sofic system defined by the finite automaton stabilized by L. The rationality and computability of the zeta function of a sofic system have been established in [Bow78] and [Man71]. The formula of computation given in [Bow78] are proved in [Béa95] by the use of a construction on finite automata called external power. The rationality of the zeta function of a rational cyclic language has been established in [BR90]. The result we give here leads to another proof and to a different computation.

We assume that the reader knows the basics of formal languages (see [Eil72]). We also assume that the reader is familiar with the elementary notions of semigroup theory. For example, notions like syntactic monoid, Green relations, regular $\mathcal{D}$-classes, minimal ideal and 0-minimal ideal are supposed to be known. We refer to [Lal79] and [Pin86] for a presentation of this subject.

The paper is organized as follows. Section 2 and 3 give the basic properties of cyclic languages and strongly cyclic languages. The chain-decomposition of a rational cyclic language in strongly cyclic languages is established in section 4. The computation of the zeta function and the generalized zeta function is done in the last section.

2 Cyclic languages

In this section, we introduce cyclic languages and give some basic properties. In the following, we denote by A a finite alphabet. In the sequel, M will always denote a finite monoid. Every element s of M has a power which is an idempotent. We denote by s^ω this idempotent.

Definition 1. A language L of A^* is said to be *cyclic* if it satisfies

$$\begin{array}{ll} \forall u \in A^*,\ \forall n > 0 & u \in L \Leftrightarrow u^n \in L \\ \forall u, v \in A^* & uv \in L \Leftrightarrow vu \in L \end{array}$$

A language is cyclic if it is closed under conjugation, power and root. If L is a submonoid of A^* which is cyclic, it is then pure [BP84].

Example 1. If $A = \{a, b\}$, the language $L = A^*aA^* = A^* - b^*$ of words having at least one a is cyclic.

Example 2. The language

$$\begin{aligned} L = \{&a^p b^{n_1} a^{n_2} b^{n_2} \dots a^{n_k} b^{n_k} a^q \mid n_i \geq 0 \text{ and } p + q = n_1\} \cup \\ \{&b^p a^{n_1} b^{n_1} a^{n_2} b^{n_2} \dots a^{n_k} b^q \mid n_i \geq 0 \text{ and } p + q = n_k\} \end{aligned}$$

is cyclic but not rational.

We will now only consider rational cyclic languages. Rational cyclic languages have the following straightforward characterization in terms of finite monoids.

Proposition 2. *Let $L \subseteq A^*$ be a rational language. Let $\varphi : A^* \twoheadrightarrow M$ be a morphism from A^* onto a monoid M such that $L = \varphi^{-1}(P)$. The language L is cyclic if and only if*

$$\begin{array}{ll} \forall s \in M,\ \forall n > 0 & s \in P \Leftrightarrow s^n \in P \\ \forall s, t \in M & st \in P \Leftrightarrow ts \in P \end{array}$$

From the previous characterization, we deduce some useful facts about the structure of the image of a rational cyclic language onto a finite monoid recognizing this language. We also deduce a property of the syntactic monoid of a cyclic language.

Corollary 3. *Let $\varphi : A^* \twoheadrightarrow M$ be a morphism from A^* onto a monoid M such that $L = \varphi^{-1}(P)$. Let H a regular $\mathcal{H}$-class of M. One has $H \subseteq P$ or $H \cap P = \emptyset$.*

Proof. Let us suppose $h_1 \in H$ belongs to P. For any $h_2 \in H$, we have $h_1^\omega = h_2^\omega = e$ where e is the idempotent of H. We then have $h_1^\omega = h_2^\omega \in P$ and $h_2 \in P$.

Corollary 4. *Let $\varphi : A^* \twoheadrightarrow M$ be a morphism from A^* onto a monoid M such that $L = \varphi^{-1}(P)$. Let H_1 and H_2 be two regular $\mathcal{H}$-classes of a regular $\mathcal{D}$-class. If one has $H_1 \subseteq P$, one also has $H_2 \subseteq P$.*

Proof. Let e_1 and e_2 be the respective idempotents of H_1 and H_2. As two idempotents of a same $\mathcal{D}$-class are conjugated (see [Pin86]), there are two elements x_1 and x_2 of M such that $x_1x_2 = e_1$ and $x_2x_1 = e_2$. If $H_1 \subseteq P$, we have $e_1 = x_1x_2 \in P$ and then $e_2 = x_2x_1 \in P$ and the $\mathcal{H}$-class H_2 satisfies $H_2 \subseteq P$ by the previous corollary.

Corollary 5. *The syntactic monoid of a cyclic language has a zero.*

Proof. Let M be the syntactic monoid of a rational cyclic language and let D the minimal ideal of M. Then D is a completely regular $\mathcal{D}$-class. By the previous corollary, we have $D \subseteq P$ or $D \cap P = \emptyset$. Let s be an element of D. For any $x, y \in M$, we have $xsy \in D$ because D is the minimal ideal of M. If we have $D \subseteq P$, the contexts of s are $M \times M$. If we have $D \cap P = \emptyset$, the contexts of s are $\emptyset \times \emptyset$. In both cases, all the elements of D are equivalent for the Nerode congruence, and D has only one element. The syntactic monoid of L has then a zero.

3 Strongly cyclic languages

We now define the notion of a strongly cyclic language.

Definition 6. Let $\mathcal{A} = (Q, A, E)$ be a deterministic finite automaton where Q is the set of states and E the set of transitions. We say that a word w *stabilizes* a subset $P \subseteq Q$ of states if we have $P.w = P$. This means

$$\begin{array}{ll} \forall p \in P & p.w \in P \\ \forall p' \in P \ \exists p \in P & p.w = p' \end{array}$$

We denote by $\mathrm{Stab}(\mathcal{A})$ the set of the words w such that w stabilizes a subset P of states in the automaton $\mathcal{A}$. It should be noticed that in this definition the subset P of states stabilized by w may depend on w. We point out that the empty word ε stabilizes the set Q and therefore belongs to $\mathrm{Stab}(\mathcal{A})$ for any automaton $\mathcal{A}$. We say that a language L is *strongly cyclic* if there is an automaton $\mathcal{A}$ such that $L = \mathrm{Stab}(\mathcal{A})$. In this case, we say that the language L *stabilizes the automaton* $\mathcal{A}$. The terminology is justified since strongly cyclic languages are cyclic. It could be proved directly but we shall obtain this fact as a consequence of the characterization of strongly cyclic languages.

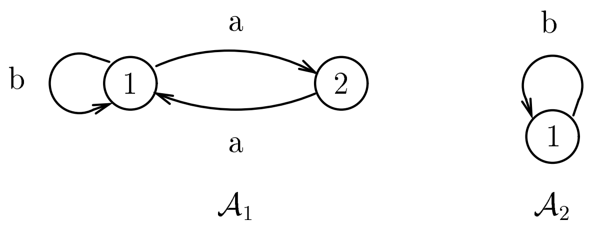

Fig. 1. Automata $\mathcal{A}_1$ and $\mathcal{A}_2$

Example 3. The languages $(b + aa)^* + (ab^*a)^* + a^*$ and b^* are respectively the strongly cyclic languages associated with the automata $\mathcal{A}_1$ and $\mathcal{A}_2$ of Figure 1.

The following result gives a characterization of the words w stabilizing a subset of states in an automaton.

Proposition 7. *Let $\mathcal{A} = (Q, A, E)$ be a deterministic finite automaton. A word w stabilizes a subset P of states in $\mathcal{A}$ if and only if there is some state q of $\mathcal{A}$ such that for any integer n, the transition $q.w^n$ exists.*

Proof. Suppose first that the word w stabilizes the subset P of states. By definition, for any state $p \in P$, the transition $p.w^n$ exists.

Conversely, suppose all the transitions $q.w^n$ exist for some state q of $\mathcal{A}$. Since the automaton is finite, there are two integers $l < m$ such that $q.w^l = q.w^m$. Let P be the set $\{q.w^i \mid l \leq i \leq m\}$. It is straightforward that the word w stabilizes the subset P.

The following theorem gives a characterization of the strongly cyclic languages.

Theorem 8. *Let L be a rational language different from A^*. The following conditions are equivalent.*

1. *The language L is strongly cyclic.*
2. *There is a morphism φ from A^* onto a monoid M having a zero such that $L = \varphi^{-1}(\{s \in M \mid s^\omega \neq 0\})$.*
3. *The syntactic monoid $M(L)$ of L has a zero and the image of L in $M(L)$ is $\{s \in M \mid s^\omega \neq 0\}$.*

Proof. Suppose first that the language stabilizes the automaton $\mathcal{A}$. Let M the transition monoid of $\mathcal{A}$ and φ the canonical morphism from A^* onto M. We show then that this monoid has a zero and that the image of L is equal to $\{s \in M \mid s^\omega \neq 0\}$. Let w be a word not belonging to L. By Proposition 7, for each $q \in Q$, there is an integer n_q such that the transition $q.w^{n_q}$ does not exist. For n greater than every n_q, the transitions $q.w^n$ do not exist for any q. The transition induced by w^n is then the empty transition and the element $\varphi(w^n)$

is a zero of the monoid M. This means that $\varphi(w)^\omega = 0$. On the contrary, for any word $w \in L$, the transition induced by w^n is not the empty transition since there is a state $q \in Q$ such that the transition $q.w^n$ exists. This proves that $\varphi(w)^\omega \neq 0$. We have then proved that L is equal to $\varphi^{-1}(\{s \in M \mid s^\omega \neq 0\})$

Suppose now there is a morphism φ from A^* onto a monoid N having a zero such that $L = \varphi^{-1}(\{t \in N \mid t^\omega \neq 0\})$. Since the morphism is onto, the syntactic monoid M of L is a quotient of N: there is morphism $\psi : N \twoheadrightarrow M$ from N onto M. The image $\psi(0)$ of the zero of N is then a zero of M. Since the zero of N does not belong to the image of L, the zero of M does not belong to the image of L. Let t a element of N such that $t^\omega = 0$. We have $\psi(t)^\omega = \psi(t^\omega) = 0$. On the contrary, if $t^\omega \neq 0$, the element t^ω belongs to the image of L, and so does $\psi(t^\omega)$. This implies $\psi(t^\omega) \neq 0$. Finally, we have $\{t \in N \mid t^\omega \neq 0\} = \psi^{-1}(\{s \in M \mid s^\omega \neq 0\})$.

Suppose that the syntactic monoid M of L has a zero and that the image of L in M is $\{s \in M \mid s^\omega \neq 0\}$. We denote by φ the canonical morphism from A^* onto M. We build the following deterministic automaton $\mathcal{A} = (Q, A, E)$. The set of states of $\mathcal{A}$ is the set $Q = M - \{0\}$ of the non-zero elements of M. The transition $q.a$ is $q.a = q\varphi(a)$ if $q\varphi(a) \neq 0$ and does not exist otherwise. It can be easily checked that a transition $q.w$ is $q.w = q\varphi(w)$ if $q\varphi(w) \neq 0$ and does not exist otherwise. We show now that the language L stabilizes the automaton $\mathcal{A}$. Let w a word in L. For $q = \varphi(w)$, the transition $q.w^n$ is $q.w^n = \varphi(w)^{n+1}$ and exists since $\varphi(w)^{n+1} \neq 0$. On the contrary, for a word w not in L, there is an integer m such that $\varphi(w)^n = 0$. The transition $q.w^n$ does not exist for any q. By the previous Proposition, the language L stabilizes the automaton L. This finishes the proof.

Corollary 9. *The previous characterization shows that strongly cyclic languages are cyclic.*

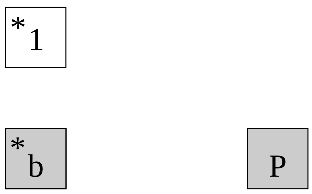

Fig. 2. Structure of the syntactic monoid of L.

The previous theorem can be used to prove that a given language is not strongly cyclic. Without this characterisation, such results are sometimes hard to obtain.

Example 4. The syntactic monoid of the language $L = A^*aA^*$ is the two elements monoid $M = \{b = 1, a = 0\}$. The $\mathcal{D}$-class structure of M is shown in Figure 2.

The image of L in M is the singleton $\{0\}$ and the previous theorem states that the language L is not strongly cyclic.

4 Decomposition of cyclic languages

In this section, we prove the main result.

By the definition of cyclic languages, a boolean combination of cyclic languages is still a cyclic language. In particular, a boolean combination of strongly cyclic languages is a rational cyclic language. The following result gives some converse.

Theorem 10. *Any rational cyclic language is a boolean combination of strongly cyclic languages.*

The proof of the theorem is based on the following lemma. By a strict quotient of M, we mean a quotient which is strictly smaller than M.

Lemma 11. *Let L be a rational cyclic language and $\varphi : A^* \twoheadrightarrow M$ a morphism from A^* onto a finite monoid M such that $L = \varphi^{-1}(P)$. Let suppose furthermore that the monoid M has a zero and that this zero does not belong to P.*

Then, either L is recognized by a strict quotient of M or there exists a strongly cyclic language L' such that $L \subseteq L'$ and such that the language $L' - L$ is recognized by a strict quotient of M.

In both cases, the zero of the quotient does not belong to the image of the language (L in the first case and $L' - L$ in the second case).

Proof. Let P the image of L in M. Suppose $D_1, \ldots, D_n$ are the 0-minimal $\mathcal{D}$-classes of M. For any $s \in D_i$ and $x, y \in M$, we have $xsy = 0$ or $xsy \in D_i$.

Suppose D_1 satisfies $D_1 \cap P = \emptyset$. For any $x, y \in M$, we have $xsy \notin P$. All the elements of D_1 are equivalent to 0 by the Nerode congruence. The language L is then recognized by the Rees quotient M/I where I is the ideal $I = D_1 \cup \{0\}$.

We can suppose that every D_i satisfies $D_i \cap P \neq \emptyset$. There exists then $s_i \in D_i \cap P$. Since $s_i \in P$, the idempotent s_i^ω is not zero and belongs then to D_i because this $\mathcal{D}$-class is 0-minimal. The element s_i belongs then to a regular $\mathcal{H}$-class H_i of D_i. Let s be an element of D_i. If $s^\omega = 0$, the element s does not belong to P because 0 does not belong to P. On the contrary, If $s^\omega \neq 0$, the element s belongs to a regular $\mathcal{H}$-class H_i' of D_i. Since $H_i \cap P \neq \emptyset$, we have $H_i \subseteq P$ and $H_i' \subseteq P$ by Corollaries 3 and 4. Finally, we have proved that

$$D_i \cap P = \{s \in D_i \mid s^\omega \neq 0\}$$

Let P' be defined by $P' = \{s \in M \mid s^\omega \neq 0\}$. By Theorem 8, the language $L' = \varphi^{-1}(P')$ is strongly cyclic. It is straightforward that $P \subseteq P'$. Since we have $D_i \cap (P' - P) = \emptyset$, the language $L' - L$ is recognized by the Rees quotient M/I where the ideal is equal to $\{0\} \cup \bigcup_{i=1}^n D_i$. This quotient is strictly smaller than M and this finishes the proof of the lemma.

The following lemma states that the class of strongly cyclic languages is closed under union and intersection.

Lemma 12. *Let L_1 and L_2 be two strongly cyclic languages. Both languages $L_1 \cup L_2$ and $L_1 \cap L_2$ are then strongly cyclic.*

Proof. We suppose that L_1 and L_2 respectively stabilizes the automata $\mathcal{A}_1 = (Q_1, A, E_1)$ and $\mathcal{A}_2 = (Q_2, A, E_2)$. We can suppose that $Q_1 \cap Q_2 = \emptyset$. The language $L_1 \cup L_2$ stabilizes then the automaton $\mathcal{A}_1 \cup \mathcal{A}_2$. The language $L_1 \cap L_2$ stabilizes the automaton $\mathcal{A}_1 \times \mathcal{A}_2 = (Q_1 \times Q_2, A, E_3)$ where the transition E_3 is defined by $(p_1, p_2).a = (q_1, q_2)$ if the transitions $p_1.a = q_1$ and $p_2.a = q_2$ exist.

We can now complete the proof of the theorem.

Proof. We prove that every cyclic language L can be written as a chain of strongly cyclic languages. This means that there are strongly cyclic languages $L_1, \ldots, L_n$ satisfying $L_1 \supseteq L_2 \supseteq \cdots \supseteq L_n$ such that

$$L = L_1 - L_2 + L_3 - \cdots \pm L_n$$

We prove the result by induction on the size of a finite monoid M having a zero and recognizing the language L. We suppose that there is a morphism $\varphi : A^* \twoheadrightarrow M$ from A^* onto a finite monoid M such that $L = \varphi^{-1}(P)$. We also suppose that the monoid M has a zero. By Corollary 5, the syntactic monoid of L has this property. If the monoid M has only one element, the language L is either $\emptyset$ or A^* which are both strongly cyclic. The empty language $\emptyset$ stabilizes the empty automaton. The full language stabilizes the automaton having one state and a transition for each letter from this state to this state.

If the zero of M belongs to the image of L in M, we replace L by $A^* - L$ which is also cyclic. It is then sufficient to prove the result for $A^* - L$. Indeed, if the complement $A^* - L$ of L can be written as a chain $A^* - L = L_1 - L_2 + \cdots \pm L_n$, the language L can be written $L = A^* - L_1 + L_2 - \cdots \mp L_n$ which is also a chain of strongly cyclic languages.

We can now suppose that the zero of M does not belong to the image P of L in M. By Lemma 11, either L is recognized by a strict quotient of M and the induction hypothesis immediately applies or there is a strongly cyclic language L' such that $L' - L$ is recognized by a strict quotient of M. By the induction hypothesis, the language $L' - L$ can be written as a chain of strongly cyclic languages, *i.e.*, $L' - L = L_2 - L_3 + \cdots \pm L_n$. The language L can be then written as the chain $L = L_1 - L_2 + \cdots \pm L_n$ where the language L_1 is equal to $L' \cup L_2$ which is strongly cyclic by Lemma 12.

We point out that the zero of the smaller monoid recognizing $L' - L$ does not belong to the image of $L' - L$. It is not necessary to replace this language by its complement any more.

We give here an example.

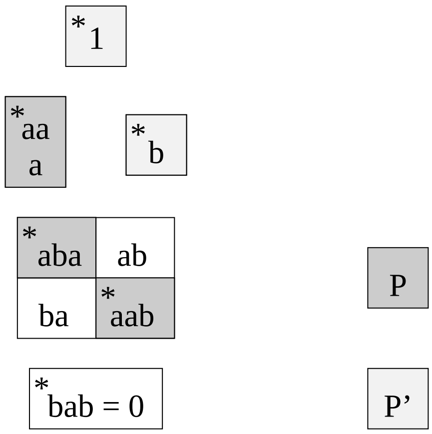

Fig. 3. Structure of the syntactic monoid of L.

Example 5. Let L be the language $(b + aa)^* + (ab^*a)^* + a^* - b^*$. The structure of the syntactic monoid of L is given on figure 3. The image P of L in $M(L)$ is equal to $P = \{a, aa, aba, aab\}$.

The subset P' defined in the proof is equal to $P' = \{1, a, aa, b, aba, aab\}$ and the language L' is $(b + aa)^* + (ab^*a)^* + a^*$. The language $L' - L$ is then equal to b^* which is strongly cyclic. The languages L' and $L' - L$ stabilize respectively the automata $\mathcal{A}_1$ and $\mathcal{A}_2$ (see Figure 1 p. 4).

We have directly proved that every rational cyclic language can be written as a chain of strongly cyclic languages. In fact, it is just necessary to prove that rational cyclic languages are boolean combination of strongly cyclic languages. A general result states that if a class $\mathcal{F}$ of sets is closed under union and intersection, every set belonging to the boolean closure of $\mathcal{F}$ can be written as a chain of sets of $\mathcal{F}$. For further details see [Car93] (chapter 3).

5 Zeta function of a cyclic language

We first give the definitions of generalized zeta function and zeta function of a language of finite words over a finite alphabet A.

If A is a finite alphabet, we note $\mathbf{Z}\langle\langle A\rangle\rangle$ (resp. $\mathbf{Z}[[A]]$) the algebra of non commutative (resp. commutative) formal series with coefficients in $\mathbf{Z}$ over the alphabet A. The subset of non commutative polynomials is denoted by $\mathbf{Z}\langle A\rangle$. We note φ the natural algebra homomorphism from $\mathbf{Z}\langle\langle A\rangle\rangle$ to $\mathbf{Z}[[A]]$ which makes the letters commute. For example, $\varphi(2ab-3ba) = -ab$. We note θ the morphism from $\mathbf{Z}[[A]]$ to $\mathbf{Z}[[z]]$ defined by $\theta(a) = z$ for each letter a of A.

Let L be a language of finite words over A, we note $\underline{L}$ the characteristic series of L. This series belongs to $\mathbf{Z}\langle\langle A\rangle\rangle$ and admits the decomposition:

$$\underline{L} = \sum_{w \in L} w = \sum_{n \geq 0} \underline{L}_n,$$

where $\underline{L}_n$ is the homogeneous part of degree n of $\underline{L}$.

Definition 13. The *generalized zeta function* of a language L over the alphabet A is now the following commutative series:

$$Z(L) = \exp\left(\sum_{n \geq 1} \frac{\varphi(\underline{L}_n)}{n}\right).$$

Definition 14. The *zeta function* of a language L over the alphabet A is the following series :

$$\zeta(L) = \theta(Z(L)) = \exp\left(\sum_{n \geq 1} \frac{a_n z^n}{n}\right),$$

where a_n is the number of words of L of length n.

It is shown for example in [Béa95] that the generalized zeta function of a strongly cyclic language L of an automaton $\mathcal{A}$ is the generalized zeta function of the sofic system defined by $\mathcal{A}$, that is the set of bi-infinite words that are labels of bi-infinite paths of $\mathcal{A}$. The zeta function of a sofic system counts periodic orbits of the symbolic dynamic system. The zeta function of a strongly cyclic language is a rational series. The computation of the generalized zeta function and the zeta function of a strongly cyclic language was done in [Béa95] by using a construction on finite automata called external power.

If $\mathcal{A} = (Q = \{1, 2, \ldots, n\}, E, T)$ is a deterministic automaton, the *external power of order* k, where $1 \leq k \leq |Q|$, of the automaton $\mathcal{A}$ is the automaton (Q', E') labelled in $\{-1, 1\} \times A$, where $Q' = \{(i_1, i_2, ..., i_k)_{1 \leq i_1 < i_2 < \cdots < i_k \leq n}\}$. There exists an edge labelled $\epsilon(\sigma)a$ from $(i_1, i_2, ..., i_k)$ to $(j_1, j_2, ..., j_k)$ if for each l with $1 \leq l \leq k$, there exists one edge in E labelled a going out from i_l and $(j_1, j_2, ..., j_k)$ is the image of $(i_1 \cdot a, \ldots, i_k \cdot a)$ by a permutation σ of signature $\epsilon(\sigma)$. The automaton $\mathcal{A}$ is equal to its external power of order 1 by identification of $+a$ to a. The *matrix associated* to an automaton labelled on $\{-1, 1\} \times A$ is the square matrix $(x_{ij})_{1 \leq i,j \leq n}$ where x_{pq} is the sum (in $\mathbf{Z}\langle A\rangle$) of the labels of edges from p to q. The commutative matrix associated is the matrix $(\varphi(x_{ij}))_{1 \leq i,j \leq n}$.

We denote by Q_i the commutative matrix associated to the external power of order i of the automaton $\mathcal{A}$. We then have (see [Béa95])

$$Z(L) = \prod_{i=1}^{n} (\det(I - Q_i))^{(-1)^i}$$

where I is the identity matrix of the same size as Q_i.

Computation of the zeta function of a cyclic language

The result of section 4 can be used to extend the previous computation of zeta functions of strongly cyclic languages to all cyclic languages. This gives an other proof of the rationality of the zeta function of a cyclic language established in [BR90]. The computation is the following:

Let L be a cyclic language. By section 4 it can be written as a chain $L_1 - L_2 + \cdots + (-1)^{r-1} L_r$, where $L_{j+1} \subseteq L_j$ for $1 \leq j \leq (r-1)$ and where all L_j are strongly cyclic languages. By definition of the generalized zeta function we have:

$$
\begin{aligned}
Z(L) &= \exp(\sum_{n \geq 1} \frac{\varphi(\underline{L}_n)}{n}) \\
&= \exp(\sum_{n \geq 1} \sum_{j=1}^{r} (-1)^{j-1} \frac{\varphi(\underline{L_j}_n)}{n}) \\
&= \exp(\sum_{j=1}^{r} (-1)^{j-1} \sum_{n \geq 1} \frac{\varphi(\underline{L_j}_n)}{n}) \\
&= \prod_{j=1}^{r} \exp(\sum_{n \geq 1} \frac{\varphi(\underline{L_j}_n)}{n})^{(-1)^{j-1}} \\
&= \prod_{j=1}^{r} (Z(L_j))^{(-1)^{j-1}}.
\end{aligned}
$$

Example 6. We compute the generalized zeta function $Z(L)$ and the zeta function $\zeta(L)$ of the cyclic language $L = (b + aa)^* + (ab^*a)^* + a^* - b^*$ introduced in example 5. The language L admits the chain-decomposition $L = L_1 - L_2$ where $L_1 = \mathrm{Stab}(\mathcal{A}_1)$ (see figure 1) and $L_2 = \mathrm{Stab}(\mathcal{A}_2)$ (see figure 1). We get:

$$
\begin{aligned}
Z(L_1) &= \frac{|1+a|}{\begin{vmatrix} 1-b & -a \\ -a & 1 \end{vmatrix}} \\
Z(L_2) &= \frac{1}{|1-b|} \\
Z(L) &= \frac{Z(L_1)}{Z(L_2)} = \frac{(1+a)(1-b)}{1-b-aa} \\
\zeta(L) &= \frac{1-z^2}{1-z-z^2}.
\end{aligned}
$$

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